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CONTENTS.

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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9. With each plane (L_1, L_2, L_3, L_4), we can associate the binary cubic $\Sigma L_r f_r$ which represents parametrically its intersections with the curve. Similarly with each point (x_1, x_2, x_3, x_4), we associate the binary cubic $\Sigma x_r F_r$ which represents parametrically the points of contact of the osculating planes through it.

The cubics associated in this manner with the points and the osculating planes of the curve are perfect cubes.

This follows from the fundamental identity

$$\Sigma f_r(t) F_r(t') \equiv k(tt')^3.$$

The condition of apolarity of the cubics associated with the point x and the plane L is clearly

$$L_1 x_1 + L_2 x_2 + L_3 x_3 + L_4 x_4 = 0,$$

remembering that $(f_r, F_s)^3 = 0$, $(f_r, F_r)^3 = (f_s, F_s)^3$, $(r \neq s)$.

Hence, *the geometrical relation of incidence of a point and a plane is equivalent to the invariantive relation of apolarity between the associated cubics.*

If the cubics associated with two points x, y are apolar, then we must have

$$\Sigma I_{rs}(x_r y_s - x_s y_r) = 0$$

that is, the points are conjugate with respect to the polarity (A′).

IV. The Quadrics containing a Twisted Cubic.

10. If, in any identical relation of the second degree between the four cubics f_1, f_2, f_3, f_4, we replace them by current point-co-ordinates x_1, x_2, x_3, x_4, we would get the equation of a quadric containing the curve. Similarly, if in an identical relation of the second degree between F_1, F_2, F_3, F_4, we replace them by current plane co-ordinates L_1, L_2, L_3, L_4, we would obtain the plane-equation of a quadric touching all the osculating planes of the curve.

Thus, from the identities (B) and (C) of II, we obtain two types (B) and (C) of quadrics containing the curve: namely, the equation of the quadrics (B) is

$$\Sigma X_r^2 [F_r(t), F_r(t)]^2 + 2\Sigma X_r X_s [F_r(t), F_s(t)]^2 = 0; \quad \ldots \text{(B)}$$

and the equation of the quadrics (C) is obtained from this by replacing t_1^2, t_1t_2, t_2^2 by arbitrary parameters λ, μ, ν respectively.

Reciprocally, from the analogous identities connecting F_1, F_2, F_3, F_4, we have two types (B′), (C′) of quadrics touching the osculating planes of the curve; the equation of (B′) would be

$$\Sigma L_r^2 \, [f_r(t), f_r(t)]^2 + 2 \Sigma L_r L_s \, [f_r(t), f_s(t)]^2 = 0; \; \dots \; (B')$$

and as before, the equation of (C′) is obtained by replacing $t_1^2, t_1 t_2, t_2^2$ in this by λ, μ, ν.

The family (C) includes all quadrics containing the curve; reciprocally the family (C′) includes all quadrics which touch the osculating planes of the curve.

For, for a quadric to contain a twisted cubic is equivalent to its passing through 7 points of the curve. Hence the equation of an arbitrary quadric containing the curve should contain *three* parameters. Since (C) contains three parameters λ, μ, ν, it follows that it represents all quadrics through the curve.

The equation (B) represents a cone which contains the curve and has its vertex at the point **t** *thereof. Reciprocally, the equation (B′) represents the conic which is the locus of the intersection of tangents to the curve with the osculating plane at the point* **t**.

For, the determinant of the quadric (B) is

$$| \, [F_r(t), F_s(t)]^2 \, | \, .$$

By multiplying the columns of this determinant by $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$ and adding, we see that this determinant vanishes in virtue of identity (D). Hence, the quadric (B) is a cone. Since (x_1, x_2, x_3, x_4), the co-ordinates of its vertex should satisfy the equations

$$\Sigma \, x_r \, (F_r, F_s)^2 = 0,$$

we see from the same identity that $x_r = f_r(t)$ $(r = 1, 2, 3, 4)$; which shews that the vertex is the point t.

It is obvious from geometry, that the family (B) includes all the cones which are contained in the family (C); in other words, a C-quadric becomes a cone if and only if the λ, μ, ν in it are replaced by t_1^2, $t_1 t_2$, t_2^2; *i.e.*, only if $(\lambda\nu - \mu^2) = 0$. But the discriminant of the C-family should be a quartic in λ μ, ν. Hence: *The discriminants of the families (C) and (C′) are perfect squares.*

We may notice one simple theorem which is an immediate consequence of this fact.

If S_1, S_2 are two quadrics of (C), $(\lambda S_1 + S_2)$ is a cone when $(a\lambda^2 + b\lambda + c)^2 = 0$ (say).

Now if the term λ^3 is missing in $(a\lambda^2 + b\lambda + c)^2$, then $b = 0$ and the term in λ will also be missing. Hence:

If S_1, S_2 be two quadrics through a twisted cubic (i.e. two quadrics having a common generator), and if there exist inscribed tetrahedra of S_1 self-polar with respect to S_2, then there also exist inscribed tetrahedra of S_2 self-polar with respect to S_1.

V. The second Transvectant of two Binary Cubics.

11. The vanishing of the third transvectant or apolar invariant of the two binary cubics associated with two points, lent itself to geometrical interpretation by means of the relations arising from the tangent linear complex of the curve.

In an analogous manner, the *second* transvectant of the binary cubics associated with two points can be interpreted by means of the family of quadrics containing the curve or touching the osculating planes thereof.

Thus, the second transvectant of the cubics associated with the points x, y is

$$(\Sigma x_r \, F_r(t), \Sigma y_r \, F_r(t))^2 = \sum_{r,s} x_r \, y_s \, (F_r(t), F_s(t))^2 \qquad \dots (1)$$

Now the cone which contains the curve and has its vertex at the point t is

$$\Sigma X^2_r \, (F_r(t), F_r(t))^2 + 2\Sigma \, X_r \, X_s (F_r(t), F_s(t))^2 = 0,$$

and the condition that the points x, y be conjugate with respect to this cone is seen to be precisely the vanishing of the expression (1). Hence:

The roots of the second transvectant of the binary cubics associated with two points x, y correspond to the vertices of the two quadric cones which contain the curve and have x, y for conjugate points.

Or, to put it otherwise:

The roots of the second transvectant of the cubics associated with x, y correspond to the extremities of the chord of the curve which is the residual intersection of all quadrics which contain the curve and have x, y for conjugate points.

Since the Hessian of a binary cubic is its second transvectant with itself, we have as a particular case:

The Hessian of the binary cubic associated with the point x corresponds to the extremities of the unique chord of the curve which passes through x.

If there exist two points P, Q conjugate with respect to *every* quadric (C), then the second transvectant of their associated binary cubics would vanish identically. Now, the twisted cubic is the complete intersection of quadrics of (C); hence it is apparent that two such points P, Q should, if they are coincident, lie on the cubic curve, and if they are distinct, should lie on a chord of the curve, harmonically separating its extremities. We have, then, the theorem:

The second transvectant $(\mathbf{f}, \mathbf{g})^2$ *of two cubics vanishes identically only in two cases; namely, when* $\mathbf{f} = \mathbf{kg} =$ *a perfect cube, and when* $\mathbf{f}$ *and* $\mathbf{g}$ *are numerical multiples of the cubicovariants of each other.*

On the Equivalence of Three Fundamental Definitions of Irrational Number

BY B. S. MADHAVARAO, M. SC.,
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Introduction.

The arithmetical theory of irrational numbers has been developed in three main forms, of which the first was given by Weierstrass in his Lectures on Analytical Functions, the second is that of Cantor[1] and the third that due to Dedekind. Of these three theories, it has been shown that those of Cantor and Dedekind are fundamentally identical[2] and it has been established that whereas the theory of Dedekind operates with the whole aggregate of rational numbers, the other operates with sequences selected out of that aggregate. We here propose to establish the fundamental identity of all the three theories of irrational number by establishing the same for the theories of Weierstrass and Cantor and to show that the entities used in the former definition are but particular types of the entities used in the latter. We further establish also the equivalence of the definitions of Weierstrass and Dedekind. Inasmuch, however, as the theory of Weierstrass does not seem to have received the notice it deserves, as is evidenced by the fact that there is no complete account of Weierstrass's theory in any of the standard English treatises on the subject, we begin by giving a brief account of this theory[3] just to the extent of serving our purpose on hand.

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[1] This theory was also independently developed by Ch. Meray in his "*Nouveau Précis d'Analyse Infinitésimale*" and Cantor's theory is practically identical with it. What Meray calls a "*variante*," Cantor calls a "*Fundamentalreihen*" and there are other slight differences of notation. However, it is to be observed that Cantor's theory as presented by Heine brings out more clearly than Meray's the distinction between the notion of generalisation of *number* and the definition of *limit*.

[2] See Hobson: *Theory of Functions of a Real Variable*; 2nd Edn., Vol. I, § 33, p. 39.

[3] A simple and clear account of Weierstrass' theory can be found in an article in the "*Encyclopédie des Sciences Mathématiques*", Tome I, Vol. I; and our account (§§ I—III and § V) is mainly based on the above.

It will also be observed that in the course of this short exposition, we have deduced some of the important properties of the class S of all real numbers, *viz.*, *seriality*, *density* and *continuity*, with the aid of Weierstrass' definition of irrational number.

Section I.

In developing Weierstrass' theory, it is not necessary to start on the hypothesis that the properties of the class of *all* rational numbers (R) are known. It is enough if we know the properties of a part of R, *viz.*, the class of absolute rational numbers (R'). Starting with this class (R'), Weierstrass defines the class of absolute real numbers (S') and then the other real numbers as couples of absolute real numbers satisfying certain laws.

Section II.

We shall make use of certain definitions :—

DEFINITION 1: Every aggregate of which we know the elements as belonging to class R', as well as the number of times that each element figures in the class, is called a "*number-magnitude*" [Ger: *zahlen-grosse*; Fr: *Quantite-numérique*]. It is further assumed that the number of times that each element appears is finite.

When such an aggregate is made up of a finite number of elements, tis regarded as equivalent to the sum of the elements.

Two such aggregates are said to be equal if the sums of their respective elements are equal. Hence it is clear that every number-magnitude with a finite number of elements is a member of the class R' and conversely every member of R' can be considered as equivalent to such a number-magnitude in more than one way.

It is however unnecessary, for the definition of such a member of R', to take into consideration, as elements of the number-magnitude, all the members of R'. It will be sufficient if only we took the integer 1 and the numbers of the type $1/n$ where n is an integer belonging to R'. Thus, for example, the aggregate $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$ denotes the rational number $(13/12)$ and (a^2, b^3, c^4) the indices denoting the number of times that a, b, c (all members of R' of the type $1/n$) occur in the aggregate, denotes the rational number $2a+3b+4c$. Hence the following definition :—

DEFINITION 2: A *number-magnitude* is said to be *absolute* if its elements are constituted by 1 and the members of R' of the type $1/n$.

We regard such an absolute number-magnitude as invariant with respect to all changes where we replace a partial aggregate of a finite number of elements by another equal to it in the sense above defined.

DEFINITION 3: We say that a rational number, a member of R′, say r, is *contained in* an absolute number-magnitude a, when we can find from a, a partial aggregate α which is equal to r.

DEFINITION 4: An absolute number-magnitude a is said to be *finite* if we can assign a member ρ of the class R′ such that every rational number r contained in a (Def. 3) should be less than ρ.

Section III.

Weierstrass' definition of irrational numbers is now very simply explained. In fact, the class of *all finite absolute number-magnitudes* is called the class of absolute real numbers. Every finite absolute number-magnitude defines an absolute real number. If such a number-magnitude has a finite number of elements, the corresponding real number is said to be rational, while all other real numbers are irrational. It is noteworthy that in this definition, there is no need as in Cantor's theory, *i.e.*, there will be no logical difficulty, in considering as identical the rational real number and the rational number defining it. In fact the definition of a rational number as an absolute number-magnitude with a finite number of elements compels us to assert this identity.

Section IV.

Let us now establish the relations of order in the class of absolute real numbers S′ thus defined. We shall hereafter denote a finite absolute number-magnitude by the same letter as the real number which it defines.

DEFINITION 5: Two members of S′ [a and b] are said to be *equal* when every member of R′ contained in the number-magnitude a is also contained in b and every member of R′ contained in b is also contained in a; in other words, *when there is a* (1,1) *correspondence between the absolute rational numbers contained respectively in the aggregates a and b.*

DEFINITION 6: *If there exists at least one absolute rational number which is contained in a without being contained in b,* the real number a is said to be greater than b and written $a > b$.

DEFINITION 7: If *there exists at least one absolute rational number contained in* b *without being contained in* a, *the* number a is said to be less than b and written $a < b$.

THEOREM 1. If a and b be any two absolute real numbers, then either $a \lesseqgtr b$.

With the above definitions the proof follows immediately. Considering, as we do, only the class S′, the question of absolute and negative numbers does not arise here.

We next proceed to deduce from Weierstrass' definition some of the properties of the class S′ and when the mode of definition of the class S from S′ is explained, it will be clear how these properties can also be easily extended to the class S.

THEOREM 2. *The class* S′ *can be arranged as a series*[1].

For,

1°. If a and b be any two elements of S′

$$a \lesseqgtr b, \text{ by the theorem stated above.}$$

2° If $a > b$, then a and b are distinct, for there is at least one member of R′ contained in a but not in b.

3°. If $a > b$ and $b > c$, then $a > c$;

for, there is at least one member of R′ contained in a but not in b. This member cannot also be contained in c, for if it were, then this same member would exist in c but not in b, *i.e.* to say, c would be greater than b which is contrary to the hypothesis. Hence it follows that $a > c$.

The theorem in question is therefore true in virtue of 1°—3°.

THEOREM 3. *The class* S′ *thus defined is dense.*

Let there be two members of S, *viz.*, a and b such that $a > b$. We have to show that there exists at least one number c (also belonging to S′) such that there is at least one absolute rational member r contained in a but not in c, and at least one r contained in c but not in b. Since b is a finite absolute number-magnitude, we can find a member ρ (of R′) such that every absolute rational number contained in b is less than ρ. Further since there is at least one member of R′, *viz.*, r in a not contained in b, we can take this ρ to be less than r in an infinity of ways. Obviously the number-magnitude defining ρ is such that $a > \rho > b$, for r is not contained in ρ and ρ is not contained in b.

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[1] In the sense in which it is used by E. V. Huntington; see his "*Continuum*," Chap I.

THEOREM 4. *The class S′ satisfies Dedekind's postulate.*

The proving of this theorem would be the same as establishing the equivalence of the definitions of Weierstrass and Dedekind which is however done in a future article. See § VIII.

Section V.

We next introduce the definition of addition of two absolute real numbers, leaving aside the definitions of multiplication, subtraction and division as not relevant for our purposes. Accordingly, the sum of two finite absolute number-magnitudes a and b is the number-magnitude c defined by the class whose elements are those which figure in a or b, each of these elements being taken a number of times equal to the number of times that it appears in a increased by the number of times it appears in b. It is further easy to see that the number-magnitude c is finite. For we can find ρ and ρ' both members of R′ so that every absolute rational number contained in a is less than ρ and every absolute rational number in b, is less than ρ'. Hence it follows that every absolute rational number contained in c is less than $\rho + \rho' = \sigma$, also a member of R′. Thus c is a finite absolute number-magnitude and the real number it defines is called the *sum* of a and b and written $c = a + b$.

Section VI.

Before proceeding to establish the equivalence of the definitions of Weierstrass and Cantor, we shall define, according to Weierstrass, the *sum* of an infinite number of finite absolute number-magnitudes. Let these last be $a, b, c,$............The sum is defined as the absolute number-magnitude s defined by the class whose elements figure in one or the other of $a, b, c,$.........each of these elements e, being taken a number of times n equal to the number of times that it figures in a, increased by the number of times it figures in b, increased by the number of times it figures in c, etc., etc. This definition is no doubt only a direct generalisation from that of the sum of two finite absolute number-magnitudes given in the previous article. But in order that this sum should have any meaning it is necessary that the number-magnitude should be finite. It should be noticed that we are here actually investigating the necessary and sufficient condition that an infinite series should have a sum. It is obvious at first that each of the numbers n should be finite as a *necessary* condition. In addition it is *necessary and sufficient* that we should be able to assign a number N such that the sum of any finite number whatever of the quantities $a, b,$...........considered should be less than N.

It is easy to notice clearly the analogy between this and the necessary and sufficient condition for the convergence of a monotonic sequence. In fact we are going to show that Weierstrass's definition is at bottom identical with the principle of convergence of a monotonic increasing sequence. Also utilising this fact we will show the identity of the definitions of Cantor and Weierstrass.

Section VII.

In order to establish the identity of the definitions of Weierstrass and Cantor, we shall show that the entities used in the former are only particular types of the entities used in the latter so that in this case there is no need, as there is in comparing the theories of Dedekind and Cantor, to show that every entity used in the latter can also be used to define the entity of the former theory. In fact consider a finite absolute number-magnitude. It is an aggregate whose members are elements of R' repeated a finite number of times in some cases. Replacing every repeated element by the rational number equivalent to it in the sense of Section II, we can assume without loss of generality that the number-magnitude is made up of the elements of R' without any repetition of elements. Since there is a serial relation among the elements of R', we can arrange the elements of our number-magnitude in the same serial order as of R' and then the number–magnitude can be written as $(u_1, u_2, \ldots\ldots\ldots, u_n, \ldots\ldots)$. Since, further, the number-magnitude is finite, we can find a number N from the class R' such that each of the elements $u_1, u_2, \ldots\ldots u_n, \ldots\ldots$(these are all the absolute rational numbers that are *contained* in the number-magnitude) is less than N. So that our number-magnitude has been reduced to *an aggregate* $(u_1, u_2, \ldots\ldots u_n, \ldots\ldots)$ *such that each element is less than the following one and all the elements are less than some fixed absolute rational number* N. But it is well-known that such an aggregate is a convergent sequence[1] in Cantor's sense; but not of the most general type. We therefore conclude that Weierstrass's "*Zahlengrösse*" is only a particular kind of Cantor's "*Fundmentalreihen.*"

With these remarks it is now clear how Weierstrass's definition comes at bottom to admit as a postulate that every infinite monotonic sequence admits an upper-bound or if we like to put it, to *create* this upper-bound when it does not exist. This *creation* of the upper-bound constitutes in fact the *definition* of the real number.

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[1] *Vide*: **Hobson**: "*Theory of Functions of a Real Variable*" 2nd Edn.; Ch. I, § 24., (p. 28.)

Section VIII.

Having now established the fundamental identity of the definitions of Weierstrass and Cantor and taking into account Hobson's proof of the equivalence of the definitions of Cantor and Dedekind, there would be no logical necessity for establishing this equivalence between the definitions of Weierstrass and Dedekind. In fact, considering a finite absolute number-magnitude as a monotonic increasing sequence of rational numbers and consequently a convergent sequence, we could reproduce Hobson's proof, above referred to, with but slight modifications. But such a proof is in its very nature artificial and we shall therefore supply a proof based on the definition of Weierstrass as independent of that of Cantor.

In order to establish the equivalence, in question, it must be shown that every finite absolute number-magnitude defines uniquely a Dedekindian cut[1] in the class R' and that this cut is the same for all number-magnitudes which represent the same real number. Conversely it must be shown that any number defined by a cut can also be represented as a finite absolute number-magnitude.

Let x be the real number which is defined by the number-magnitude a. Since this number-magnitude is finite (Def. 4, § II) we can assign a member ρ of the class R' such that every rational number r contained in a should be less than ρ. A section of the rational numbers belonging to R' can now be defined as follows :—Let *any* number ρ which is greater than *every* rational number contained in a, be put in the R_2-class and let every other member of R' be placed in the R_1-class. That this cut is Dedekindian is seen by observing that neither R_1 nor R_2 is a null class and further that every member of R_1 is less than every member of R_2; for, a member of R_1 is either contained in the number-magnitude a or is less than at least *one* such rational number as otherwise it would belong to R_2. Hence in either case such a member of R_1 is less than every member ρ of R_2 as above defined.

Next, let b be another number-magnitude which defines the same real number x. We have to show that the section corresponding to b is identical with that which corresponds to a. Let (R_1', R_2') be the section corresponding to b. Since a and b define the same real number it follows that (See Def. 1. § IV) every absolute rational number contained

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[1] This is virtually the theorem 4 in § IV and thus, in proving this first part we shall be proving that theorem also.

in a is also contained in b and every such number contained in b is also contained in a. Thus a member ρ of the class R_2 being greater than every rational number contained in a is also greater than every rational number contained in b; so that ρ must also belong to R_2'. Similarly it can be proved that every number which belongs to R_2' must also belong to R_2. Thus, there is a (1, 1) correspondence between the elements of the upper-classes of the two sections under consideration; but this is one of the cases where we can assert the equality of two irrational numbers as defined by sections[1]. We have therefore shown that the section (R_1, R_2) is identical with the section (R_1', R_2').

Finally to show that a number-magnitude can be found so as to define the real number corresponding to a given section (R_1, R_2) let us assume, in order to exclude trivial cases, that neither R_1 has a greatest rational number nor R_2 a least one. Let us choose an aggregate a whose elements are all rational numbers of the class R_1 with each element repeated a finite number of times, if necessary. Such an aggregate would be a number-magnitude. Let us further choose this number-magnitude a so as to satisfy the following conditions :—

1°. If r be *any* rational number contained in a, there is at least one member of R_1, also a rational number which is greater than r.

2°. *Every* rational number r contained in a should be less than *any* member ρ (also an absolute rational number) of the class R_2.

In virtue of the condition 2°, it follows that the number-magnitude a is *finite* (Def. 4. § II) and hence defines a real number. We shall now show that this real number is identical with the real number (R_1, R_2). Let b and c be any two rational numbers, belonging respectively to the classes R_1 and R_2, considered as number-magnitudes with a finite number of elements. Since every r contained in a is less than any rational member of the class R_2, it is easy to see that there exists at least one absolute rational number which is contained in c (for example, the rational number c itself) without being contained in a. We can therefore write (See Def. 2, § IV)

$$c > a \qquad \ldots \qquad \ldots \quad (1)$$

[1] For the different cases where we can assert this equality, see **Tannery**: *Leçons sur la théorie des fonctions d'une variable réelle.* (1re Edition); t. I; Ch. I.

Further, in virtue of condition 1°, it follows that there is at least one rational number, a member of R_1, which is greater than the rational number b. We can therefore find at least one rational number r so as to be contained in a but which should not be contained in b and thus (Def. 3, § IV) we can write

$$b < a \qquad \ldots \qquad \ldots \quad (2)$$

Since we have supposed at the outset that there is neither a greatest rational number in the R_1-class nor a smallest one in the R_2-class, the conclusions in (1) and (2) show that the number-magnitude a defines the number (R_1, R_2)[1].

The complete equivalence of the two theories of Dedekind and of Weierstrass has now been established.

Section IX.

From what has been established above, it is clear that whereas to a given number-magnitude there corresponds an unique section, we can find an infinite number of number-magnitudes corresponding to a given section. Dedekind's theory operates with the whole class R, while that of Weierstrass operates with particular types of parts of R and in this respect Weierstrass's theory is analogous to that of Cantor. This is also otherwise evident since we have already proved that a number-magnitude is a particular type of convergent sequence.

We might say, in conclusion, that there are only two *independent* theories of irrational number, *viz*., the *Méray-Weierstrass-Cantor* definition and the *Dedekind* definition[2].

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[1] A simpler proof of this can be obtained by looking at a number-magnitude as a particular type of convergent sequence, as defined by Cantor. In fact the convergent sequence $\{ x_{2n-1} \}$ used by **Hobson** ; *Theory of Functions of a Real Variable*, 2nd Ed., Vol. 1, § 34, p. 40 would be a number-magnitude satisfying our conditions and defining (R_1, R_2).

[2] A comparison of the three theories is made by **Cantor** himself. See *Math. Ann.*, Vol. 21.

Also see, the article "*Irrationalzahlen und Konvergenz Unendlicher Prozesse*" by **Pringsheim** in the *Encycl. der. Math. Wiss.*, I. A. 3, pp 54—7.

The Converse of Fermat's Theorem

By Prof. N. B. Mitra, M. A.

1. We know that if n is prime and a prime to n, then

$$a^{n-1} \equiv 1 \pmod{n}.$$

This is known as Fermat's Theorem ... (A)

The question is; Does this theorem hold good when n is composite and if so, for what values of n? It is proposed in this note to indicate methods of solving the congruence $a^{n-1} \equiv 1 \pmod{n}$, where n is composite and a prime to n.

2. We shall require the help of the following theorem:—

If $x \equiv a \pmod{p}$ and $x \equiv a \pmod{q}$, where p and q are prime to each other, than $x \equiv a \pmod{pq}$.

For $x = lp + a = mq + a$. Therefore $lp = mq$. Since q does not divide p, it must divide l, so that $l = kq$. Hence $x = k\,pq + a$, or

$$x \equiv a \pmod{pq} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B)}$$

3. First let us suppose n to consist of only two prime factors p and q of which $p < q$. We shall prove the following theorem:—

If $q \equiv 1 \pmod{z}$, where z is the haupt exponent of a, mod. p and $p \equiv 1 \pmod{x}$, where x is the haupt exponent of a, mod. q, then

$$a^{pq-1} \equiv 1 \pmod{pq}.$$

For, since $q \equiv 1 \pmod{z}$, $q = kz + 1$, where k is an integer. ... (1)

Similarly $\quad p = lx + 1 \quad \ldots \quad \ldots \quad \ldots$ (2)

Also $\quad a^z \equiv 1 \pmod{p}$ and $a^x \equiv 1 \pmod{q}$. ... ... (3)

By (A), $a^{p-1} \equiv 1 \pmod{p}$. Raising to the qth power and multiplying both sides by a^{q-1}, we have

$$\begin{aligned} a^{pq-1} &\equiv a^{q-1} \pmod{p} \\ &\equiv a^{kz} \quad \text{,,} \quad \text{by (1)} \\ &\equiv 1. \quad \text{,,} \quad \text{by (3)} \quad \ldots \quad (4) \end{aligned}$$

Similarly

$$\begin{aligned} a^{q-1} &\equiv 1 \pmod{q} \text{ by (A)} \\ \therefore \quad a^{pq-1} &\equiv a^{p-1} \quad \text{,,} \\ &\equiv a^{lx} \quad \text{,,} \quad \text{by (2)} \\ &\equiv 1. \quad \text{,,} \quad \text{by (3)} \quad \ldots \quad (5) \end{aligned}$$

From (4) and (5), by (B),

$$a^{pq-1} \equiv 1 \pmod{pq}.$$

4. To find numerical solutions for an assigned valve of a, we proceed thus. Give any value to p; then (1) gives us the corresponding linear form of q. Take any of the values of q given by this linear form and find the corresponding haupt exponent x. If this is a divisor of $p-1$, then $n = pq$ is a solution of the problem.

Ex. Let $p = 17$; then $q = 16k + 1$ and we get the following values of x for different values of a corresponding to successive values of $q > p$ and $\not> 1000$.

Values of q	Values of x for a equal to														
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
97	48	48	24	96	12	96	16	24	96	48	16	96	96	96	12
113	28	112	14	112	112	14	28	56	112	56	112	56	28	4	7
193	96	16	48	192	96	24	32	8	192	64	24	64	32	192	24
241	24	120	12	40	20	240	8	60	30	48	120	240	240	3	6
257	16	256	8	256	256	256	16	128	256	64	256	128	256	32	4
337	21	168	21	112	56	56	7	168	336	112	168	21	168	336	21
353	88	352	44	352	32	32	88	176	32	88	352	352	352	176	22
401	200	400	100	25	400	200	200	200	200	200	400	400	25	400	50
433	72	27	36	432	216	432	24	27	432	216	108	216	432	432	18
449	224	448	112	14	448	112	224	224	32	56	448	448	224	448	56
577	144	48	72	576	72	576	48	24	576	96	144	576	192	576	36
593	148	592	74	592	592	592	148	296	592	592	592	592	592	74	37
641	64	640	32	64	640	320	64	320	32	160	640	20	160	640	16
673	48	168	24	672	336	112	16	84	224	672	28	84	168	224	12
769	384	48	192	128	384	256	128	24	192	768	64	768	768	384	96
881	55	880	55	440	880	80	55	440	440	440	880	440	880	880	55
929	464	928	232	232	928	928	464	464	464	464	928	928	928	928	116
977	488	976	244	976	976	488	488	488	976	488	976	976	61	488	122

We pick out from this table the values of x which divide 16 and we get the following solutions for $p=17$, $q \not> 1000$, $a \not> 20$:—

$q = 97$ for $a=8$, 12 and 18; $q=113$ for $a=15$ and 18; $q = 193$ for $a=3$ and 9; $q = 241$ for $a=8$; $q = 257$ for $a = 2$, 4, 8 and 16; $q = 401$ for $a=20$; $q=641$ for $a=16$; $q=673$ for $a=8$; $q=929$ for $a=18$;

Also $q=337$, 353, 433, 449, 577, 593, 769, 881, 977 for no values of a within the assigned limits.

5. Another method of getting numerical solutions for assigned values of a is as follows :—

Take any particular value of p; calculate z, the exponent to which a appertains, mod. p, so that $a^z \equiv 1 \pmod{p}$ ... ... (6)

Then $p-1 = kz$, by (A) ... ... (7)

Factorize $a^{p-1}-1$. If q be a prime factor which is of the form $mz+1$ then $n = pq$ is a solution of the problem.

$$\text{For,}\quad a^{q-1} = a^{mz} \equiv 1 \pmod{p} \text{ by (6)}$$

$$\therefore\quad a^{pq-1} \equiv a^{p-1} \quad ,,$$

$$\equiv 1 \quad ,, \qquad \ldots \qquad \ldots \quad (8)$$

$$\text{Again}\quad a^{q-1} \equiv 1 \pmod{q}$$

$$\therefore\quad a^{pq-1} \equiv a^{p-1} \quad ,,$$

$$\equiv 1 \quad ,, \text{ since } q \text{ divides } (a^{p-1}-1). \quad (9)$$

From (8) and (9), by (B),

$$a^{pq-1} \equiv 1 \pmod{pq}.$$

Ex. Let $a = 2$, $p = 47$. Now

$$2^{46}-1 = 3.2796203.47.178481:$$

$$\text{Here}\quad z = 23 \text{ and } \left.\begin{array}{r} 2796203 \equiv 1 \\ 178481 \equiv 1 \end{array}\right\} \pmod{23}.$$

Hence $q = 178481$ and 2796203 are solutions of the problem.

6. We give below tables of solutions for (1) $a = 2$; $p \ngtr 136$; (2) $a=3$; $p \ngtr 82$; $a = 5, 6, 7, 10$; $p \ngtr 59$.

p	$a = 2$	$a = 3$	$a = 5$
	q	q	q
7	—	13	31
11	31	61	71, 521
13	—	73	601
17	257	193	11489
19	73	37, 757	829, 5167
23	89, 683	67, 661, 3851	67, 5281, 12207031
29	113	1093, 16493	449, 234750601
31	151, 331	61, 271, 4561	61, 181, 1741, 7621
37	73, 109	73, 757, 530713	829, 6597973

41	61681	42521761	241, 521, 9161, 632133361
43	127, 337, 5419	547, 1093, 2269, 368089	127, 378, 7603, 19531, 519499
47	178481, 2796203	1001523179, 23535794707	...
53	157, 1613	398581, 797161, 4795973261 *	203450521*
59	233, 1103, 2089 3033169	523, 6091, 28537 5385997, 20381027	...
61	1321	271, 47763361	
67	20857, 599479	661, 25411, 176419 2413941289	
71	281, 86171, 122921	2664097031 *, 374857981681	
73	433, 38737	6481, 530713.	

p	$a = 2$ (concld.)
	q
83	13367, 164511353, 8831418697
89	353, 2113, 2931542417
97	193, 673, 65537, 22253377
101	601, 1801, 8101, 268501
103	307, 2143, 2857, 6529, 11119, 131071
107	6361, 69431, 20394401, 28059810762433
109	246241, 262657, 279073
113	5153, 15790321, 54410972897
127	5419, 92737, 649657, 77158673929
131	2731, 8191, 409891, 7623851, ...

p	$a = 6$ q	$a = 7$ q	$a = 10$ q
3	—	—	11
5	37	—	101
7	31, 43	—	13, 37
11	101, 311	191, 2801	41, 271, 9091
13	37, 97	181	37, 9901
17	1297, 98801	1201, 169553	5882353
19	2467, 46441	37, 1063, 117307	37, 52579, 333667
23	3154757, 51828151	1123, 293459, 10746341	4093, 8779, 21649, 513239
29	197, 421, 5030761	113, 4733, 13564461457	281, 4649, 121499449
31	1171, 1201, 1950271	159871, 6568801	211, 241, 2161, 9091, 2906161, 271
37	73, 541, 55117, 46441	181, ...	9901, 999999000001 *
41	241, ...	281, 1201, 2801, 4021 n	27961, 9999000099990001 *
43	55987, ...	51031, 309079, ...	127, 1933, 2689, 459691 909091, 10838689
47	139, 3221		139, 2531, 549797184491917 *
53	937, ...	157, ...	521, 265371653, 1058313049
59	...	...	3191, 16763, 43037, 62003, 77843839397 *

Note : A dash — indicates that there are no solutions.

Dots ... indicate that there may possibly be solutions which the author of the note has not succeeded in finding.

An asterisk * indicates that the composition of the number is not known to the writer.

7. In the *Messenger of Mathematics*, Vol. XXVII, (p. 174), Mr. J. H. Jeans gives a method of solving the congruence

$$2^{n-1}-1\equiv 0 \pmod{n},$$

n being composite. For the case of two prime factors of n he adds "for values of p from 3 to 31, the only solutions are $n=341$, 1387, 4369, 4681, 10261." It will be seen from the above tables that besides the five given by Mr. Jeans there are three other solutions, namely, $n=2047$, 3279, 15709.

8. Next let $n=pqr$, where p, q, r are three different primes of which $p<q$ and $r>pq$ and a prime to each of them.

Suppose we have found p, q by either of the foregoing methods, such that $a^{pq-1}\equiv 1 \pmod{pq}$. Let x, y be the least exponents of a which are congruent to 1, moduli r and pq respectively. Then if

$$pq \equiv 1 \pmod{x} \text{ and } r\equiv 1 \pmod{y},$$

we shall have

$$a^{pqr-1}\equiv 1 \pmod{pqr}.$$

The proof is exactly similar to that of Art. 3. To find r, therefore, we proceed thus: having found the values of p and q, we first calculate y; then we know that r is of the linear form $my+1$. Take any particular value of r of this linear form and calculate x; then, if x divides $pq-1$, we have obtained a solution $n=pqr$.

Or, we may factorise $(a^{pq-1}-1)$ and take for r any prime factor of this which is of the form $my+1$.

9. It is possible, however, to find a solution even when the congruence

$$a^{pq-1}\equiv 1 \pmod{pq}$$

is not satisfied.

For, if x and y have the same meaning as in Art. 8, then

$$a^{pqr-1}\equiv 1 \pmod{pqr},$$

if $$pq\equiv 1 \pmod{x} \qquad \ldots \qquad \ldots \qquad (10)$$

and $$pqr\equiv 1 \pmod{y}. \qquad \ldots \qquad \ldots \qquad (11)$$

Proof: $a^{r-1}\equiv 1 \pmod{r}$ by (A).

Raise to the pq^{th} power and multiply both sides by a^{pq-1}; we have

$$a^{pqr-1}\equiv a^{pq-1} \pmod{r}\equiv a^{lx} \pmod{r} \text{ by (10)}$$

$$\equiv 1 \quad ,, \quad , \text{ since } a^{x}\equiv 1 \pmod{r}$$

Also $$a^{pqr-1}\equiv a^{ay} \text{ by (11)}$$

$$\equiv 1 \pmod{pq}.$$

$$\therefore \quad a^{pqr-1}\equiv 1 \pmod{pqr}, \text{ by (B)}.$$

Ex. Let $a = 2, p = 3, q = 5$. Then $y = 4$. We have to solve

$$15r\equiv 1 \pmod{4}$$

and we get $r = 4m-1$. Now x divides $r-1$; therefore r is of the linear form $lx+1$. But by (10) x divides $pq-1$ or 14. Hence x is 2, 7 or 14; and r is of one of the linear forms $2l+1$, $7l+1$ or $14l+1$. Since r is a prime, the second of these forms is included in the third. Combining these forms with the form $4m-1$, we get two solutions: namely, $r=43$ and 127.

Or, we may proceed thus: since $r=4m-1$, we get

$$r=19,\ 23,\ 31,\ 43,\quad \ldots\quad 127,$$

and the corresponding values

$$x=18,\ 11,\ 5,\ 14,\quad \ldots\quad 7,$$

whence as before $\quad r=43$ or 127.

Or again, we may factorise $2^{11}-1$; take as possible values of r, the prime factors of this; find the corresponding values of x; and pick out those which are of the linear form given by (11).

10. In the particular case $q=mz+1$, where z is the least exponent for which $a^z\equiv 1 \pmod{p}$, it can be shown that

$$q\equiv 1 \pmod{y}.$$

For, $a^{q-1}=a^{mz}\equiv 1 \pmod{p}$ and $a^{q-1}\equiv 1 \pmod{q}$ by (A).

Hence $\quad a^{q-1}\equiv 1 \pmod{pq}$.

But $\quad a^y\equiv 1 \pmod{pq}$.

$\therefore \quad y$ divides $q-1$, or $q\equiv 1 \pmod{y}$.

Hence condition (11) reduces in this case to $pr\equiv 1 \pmod{y}$.

11. We give below a table of solutions in the case $a=2$ for
(1) $p=3$, $q<59$; (2) $p=5$, $q<31$; (3) $p=7$, $q<19$.

In the paper referred to in Art. 7, Mr. Jeans says: "The general case of more than two prime factors, up to $p=7$ gives the single solution $n=645$." It will be seen from the tables that there are many other solutions in the case of three prime factors, *viz.*, 1905, 8481, 12801, *etc.*; and there are *certainly* other solutions for more than three factors.

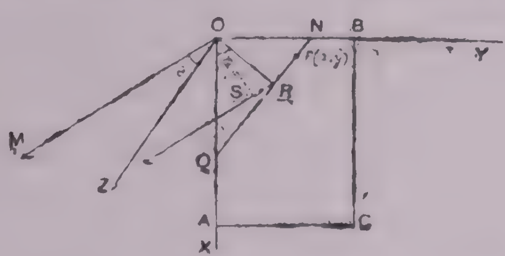
$p=3$		$p=5$		$p=7$	
q	r	q	r	q	r
5	43, 127	7	43691	13	151, 331, 631, 23311
11	257, 65537	13	257, 65537	17	179951, 3203431780337*
17	251, 4051	19	2351	19	20857, 312709
31	277, 30269	23	1212847		
53	2687	29	241, 577		

Diffraction by a Rectangular Aperture

By S. Srinivasa Rau.

The diffraction pattern produced at infinity by a rectangular aperture in parallel light has been already investigated (Lord Raleigh's *Scientific Papers*, Vol. III, p. 80, or R.W. Wood's *Physical Optics*, 1911 Edn., p. 195). In that investigation, however, an assumption is made, namely, that the effective portion of the wave is confined to a very small portion at the centre and consequently the effect due to obliquity is neglected. As will be seen below, this problem can be solved more generally.

The usual way of attacking this problem is to consider the plane wave to be changed by a condensing lens into a convergent spherical wave and then to find out the diffraction pattern produced by the aperture in the focal plane of the condensing lens. A rigorous solution has not been obtained in this way because it involves very complicated calculations. It does not matter, as far as the diffraction pattern is concerned, whether the aperture is placed just behind or just in front of the condensing lens. But, the calculations are very much simplified if we consider the aperture to be situated in front of the condensing lens.



Let OACB be the rectangular aperture of length a (= OA) and breadth b (= OB). Suppose it to be placed at right angles to the plane wave front. Take OA and OB as the axes of x and y and OZ in the direction of propagation of the wave. Consider the light diffracted in the direction OM making an angle θ with OZ. Let the plane MOZ make an angle ϕ with OX. If, from any point P (x, y) within the rectangle, a straight line NPQ be drawn perpendicular to the plane MOZ, that is, at right angles to OR, where $X\hat{O}R = \phi$, it can be easily seen that the dif-

fracted rays from all points on this straight line start with the same phase of vibration, but have a path difference from that starting from O equal to RS ($=$OR sin θ); hence the phase difference is equal to $\frac{2\pi}{\lambda}\cdot$OR sin θ. Taking the element of area at P to be $dxdy$, the resultant intensity I due to the whole rectangle at infinity is equal to

$$K\left[\left\{\int_0^a\int_0^b \cos\frac{2\pi \text{OR} \sin\theta}{\lambda}\,dxdy\right\}^2+\left\{\int_0^a\int_0^b \sin\frac{2\pi\,.\,\text{OR}\sin\theta}{\lambda}\,dxdy\right\}^2\right]$$

where K is a constant depending upon the amplitude of the initial vibration and the obliquity θ of diffraction. For a given value of θ, K is the same for *all points* within the rectangle. Hence it is taken out of the sign of integration in the above expression.

Now $\qquad\qquad$ OR$=x\cos\phi + y\sin\phi$.

$$\therefore\ I = K\left[\left\{\int_0^a dx\int_0^b \cos\frac{2\pi\,(x\cos\phi+y\sin\phi)\sin\theta}{\lambda}\,dy\right\}^2 + \left\{\int_0^a dx\int_0^b \sin\frac{2\pi\,(x\cos\phi+y\sin\phi)\sin\theta}{\lambda}\,dy\right\}^2\right]$$

$$= K\left[I_1^2+I_2^2\right]\ \text{(say)};$$

$$I_1=\int_0^a dx\left[\frac{\lambda}{2\pi\sin\phi\sin\theta}\sin\frac{2\pi\,(x\cos\phi+y\sin\phi)\sin\theta}{\lambda}\right]_0^b$$

$$=\frac{\lambda}{2\pi\sin\phi\sin\theta}\int_0^a\left\{\sin\frac{2\pi\,(x\cos\phi+b\sin\phi)\sin\theta}{\lambda} - \sin\frac{2\pi\,(x\cos\phi)\sin\theta}{\lambda}\right\}dx$$

$$=\frac{\lambda^2}{4\pi^2\sin^2\theta\sin\phi\cos\phi}\left[-\cos\frac{2\pi\,(x\cos\phi+b\sin\phi)\sin\theta}{\lambda} + \cos\frac{2\pi\,x\cos\phi\sin\theta}{\lambda}\right]_0^a$$

$$=\frac{\lambda^2}{4\pi^2\sin^2\theta\sin\phi\cos\phi}\left[-\cos\frac{2\pi\,(a\cos\phi+b\sin\phi)\sin\theta}{\lambda} + \cos\frac{2\pi\,a\cos\phi\sin\theta}{\lambda}+\cos\frac{2\pi\,b\sin\phi\sin\theta}{\lambda}-1\right]$$

$$=\frac{\lambda^2}{4\pi^2\sin^2\theta\sin\phi\cos\phi}\left[\cos\alpha+\cos\beta-\cos(\alpha+\beta)-1\right],$$

where $\alpha = \frac{2\pi\ a\cos\phi\sin\theta}{\lambda}$ and $\beta = \frac{2\pi b\sin\phi\sin\theta}{\lambda}$.

The expression within the square brackets may be transformed into

$$4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\alpha + \beta}{2}.$$

$$\therefore \; I_1 = \frac{\lambda^2}{\pi^2 \sin^2 \theta \sin \phi \cos \phi} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\alpha + \beta}{2}.$$

Similarly

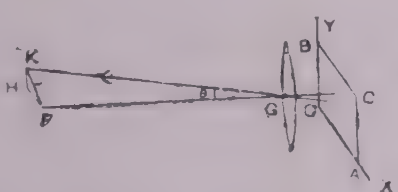
$$I_2 = \frac{\lambda^2}{\pi^2 \sin^2 \theta \sin \phi \cos \phi} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\alpha + \beta}{2}.$$

$$\therefore I = K\,(I_1^2 + I_2^2)$$

$$= K \frac{\lambda^4}{\pi^4 \sin^4 \theta \sin^2 \phi \cos^2 \phi} \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2}$$

$$= K\, a^2 b^2 \left\{ \frac{\sin \dfrac{\pi\, a \cos \phi \sin \theta}{\lambda}}{\dfrac{\pi\, a \cos \phi \sin \theta}{\lambda}} \right\}^2 \left\{ \frac{\sin \dfrac{\pi\, b \sin \phi \sin \theta}{\lambda}}{\dfrac{\pi\, b \sin \phi \sin \theta}{\lambda}} \right\}^2 \quad \ldots \quad \text{(A)}$$

We must next determine at what point in the focal plane of the condensing lens these diffracted rays are brought to a focus. Suppose the lens is placed so that its axis is perpendicular to OACB.



Through G, the optical centre of the lens, draw GK parallel to the diffracted light, cutting the focal plane at K. It is clear that K is the point at which the diffracted light is brought to a focus. Draw GF, the axis of the lens, and FH and HK parallel to OX and OY. The angles FGK and KFH are seen to be equal to θ and ϕ respectively.

Let FH and HK, representing the co-ordinates of the point K relative to F, be denoted by ξ and η. Then

$$\xi = \text{KF} \cos \phi = f \tan \theta \cos \phi,$$

and

$$\eta = \text{KF} \sin \phi = f \tan \theta \sin \phi;$$

where f is the focal length of the lens.

$$\therefore \quad \cos\phi \sin\theta = \frac{\xi \cos\theta}{f}; \text{ and } \sin\phi\sin\theta = \frac{\eta\cos\theta}{f}.$$

Equation (A) is accordingly transformed into

$$I = K\, a^2\, b^2 \left\{ \frac{\sin \dfrac{\pi\, a\, \xi \cos\theta}{\lambda f}}{\dfrac{\pi\, u\, \xi \cos\theta}{\lambda f}} \right\}^2 \times \left\{ \frac{\sin \dfrac{\pi\, b\, \eta \cos\theta}{\lambda f}}{\dfrac{\pi\, b\, \eta \cos\theta}{\lambda f}} \right\}^2 \quad \ldots \text{(B)}$$

(B) is the full expression for the intensity of illumination at any point in the focal plane of the condensing lens.

If ξ and η are negligible when compared to f, $\cos\theta$ may be written equal to 1. Then (B) is reduced to the expression which has been fully treated in Wood's *Physical Optics*, 1911, p. 197.

Notes and Questions.

Normals from a point to a Quadric in N dimensions.

I. Using Cartesian co-ordinates, the equation to a quadric in N dimensions may, in general, be reduced to

$$a_1 x_1^2 + a_2 x_2^2 + \ldots\ldots\ldots + a_n x_n^2 = 1. \qquad \ldots \quad (1)$$

Let P be any point $(\alpha_1, \alpha_2, \ldots \alpha_n)$; consider the locus of points Q whose polar planes with respect to (1) are perpendicular to PQ.

Since the direction cosines of PQ are $x_1 - \alpha_1$, $x_2 - \alpha_2$ etc., the equations to the locus of Q are

$$\frac{x_1 - \alpha_1}{a_1 x_1} = \frac{x_2 - \alpha_2}{a_2 x_2} = \ldots\ldots = \frac{x_n - \alpha_n}{a_n x_n} = t \text{ (suppose)};$$

so that
$$x_r = \frac{\alpha_r}{1 - a_r t}. \qquad (r = 1, 2 \ldots n) \qquad \ldots \quad (2)$$

If we replace the Cartesian co-ordinates by homogenous co-ordinates, (2) shows that the $n + 1$ co-ordinates of any point on the locus are proportional to polynomials of the nth degree in t.

The locus, therefore, cuts any plane in n points and is thus *a twisted n-ic.*

This n-ic intersects the quadric (1) in $2n$ points which are the feet of the normals from P.

Further the n-ic passes

(1) through the centre, corresponding to $t = \infty$;

(2) through P itself, corresponding to $t = 0$;

(3) through the points at ∞ on the n axes of the quadric, corresponding to $t = \dfrac{1}{a_r}$.

An important property of this n-ic is that *any region of r-dimensions which cuts the n-ic in* $(r + 1)$ *points*, (such a region will be called a focal region of the n-ic), *is normal to its polar region with respect to the quadric.*

For, let the region of r dimensions cut the n-ic in $Q_1, Q_2, \ldots\ldots Q_{r+1}$.

The polar region of $Q_1 Q_2 \ldots Q_{r+1}$ is the common region of the polar planes of $Q_1, Q_2, \ldots Q_{r+1}$.

But the polar plane of Q_k is perpendicular to PQ_k.

Hence the polar region of $Q_1Q_2 \ldots Q_{r+1}$ is perpendicular to the region determined by $PQ_1, PQ_2 \ldots PQ_{r+1}$, *i.e.* to the $(r+1)$-dimensional region $PQ_1 Q_2 \ldots Q_{r+1}$.

Hence it is perpendicular to every r-dimensional region contained in $PQ_1Q_2 \ldots Q_{r+1}$, in particular to the region $Q_1Q_2 \ldots Q_{r+1}$.

Another obvious property of the n-ic is that it has figures inscribed in it self-conjugate for the quadric; for, there is one such figure, *viz.*, $O\infty_1\infty_2 \ldots \infty_n$, where O is the centre and $O\infty_r$ an axis, and therefore there is an infinite number.

In particular, we note that the n-ic cuts the plane at ∞ in n points which form a self-conjugate set with respect to the imaginary quadric at ∞ (which is the locus of the circular points in every two-dimensional region in the space). The n-ic has thus the same relation to our n-space, as the rectangular hyperbola bears to space of two dimensions.

II. A chord of the n-ic is, as we have seen, normal to its polar region. The totality of lines normal to their polar regions forms a complex which is such that: (1) ∞^1 lines of the complex pass through any point, (2) all these lines are generators of a cone of the $(n-1)$th degree.

The lines normal to their polar regions, through the point P defining the n-ic, are the chords of the n-ic through P and their locus is the cone with vertex at P containing the n-ic, a cone of the $(n-1)$th degree, as it ought to be.

Similar propositions hold with regard to regions of higher dimensions normal to their polar regions.

III. Another property of the n-ic, which is interesting as an extension of a property in two dimensions, is :—If the polar plane of any point Q on the n-ic cuts the axis in $a_1, a_2 \ldots\ldots a_n$, then the hyperspheres $Oa_1, a_2 \ldots\ldots a_n$ have a common radical plane which is the plane through the centre parallel to the plane through the feet of the perpendiculars from P on the axes.

The proof is left to the reader.

IV. The twisted n-ic defined by P has been shown to pass only through $n+2$ points, *viz.* the point P itself, the centre O and the n points at ∞ on the axes.

But a n-ic requires $n+3$ points for its specification.

We may expect therefore that the n-ic passes through some other point specially related to P. The clue to this point is found by taking a projective definition of the normal.

Let S be Cayley's *Absolute Quadric* and T a given quadric. Using homogenous co-ordinates and taking the common self-conjugate figure as the figure of reference, the equations to these quadrics are

$$\Sigma\, b_r\, x_r^2 = 0,$$
$$\Sigma\, a_r\, x_r^2 = 0. \qquad (r = 0, 1, 2 \ldots\ldots n)$$

Let P be $(\alpha_0, \alpha_1, \ldots\ldots \alpha_n)$, and let Q $(\beta_0, \beta_1, \ldots\ldots {}_n)$ be a point such that its polar plane with respect to T is perpendicular to PQ in the projective sense.

The equations to PQ are

$$\frac{X_0 - \alpha_0}{\beta_0 - \alpha_0} = \frac{X_r - \alpha_r}{\beta_r - \alpha_r} = \lambda,$$

so that $$X_r = \lambda\beta_r + (1 - \lambda)\alpha_r.$$

The polar plane of Q is $\Sigma\, a_r\, \beta_r X_r = 0$ and its pole with respect to the Absolute is

$$\left(\frac{a_0\beta_0}{b_0}, \ldots\ldots \frac{a_n\beta_n}{b_n}\right).$$

The condition that this point lies on PQ is

$$\frac{b_r\{\lambda\beta_r + (1-\lambda)\alpha_r\}}{a_r\beta_r} = t,$$

or $$\beta_r = \frac{b_r\,\alpha_r}{a_r - \mu b_r}, \text{ where } \mu = \frac{\lambda}{t}.$$

These are then the equations to the n-ic which is the locus of Q.

It is at once verified that the n-ic passes through the $(n+1)$ corners of the self-conjugate figure and through the point P itself corresponding to $\mu = \infty$.

But there is one additional point corresponding to $\mu = 0$, on the locus,

viz., $$\left(\frac{b_r\,\alpha_r}{a_r}\right).$$

This point is the pole with respect to T of the polar plane of P with respect to the Absolute.

In the Euclidean case first considered this last point coincides with one of the vertices of the self-conjugate figure, *viz.*, the centre of T.

In this case therefore the n-ic must be specified by the fact that it has a special tangent at O, the equations to which in Cartesian co-ordinates already employed will be found to be

$$\frac{X_1 a_1}{\alpha_1} = \frac{X_2 a_2}{\alpha_2} \ldots\ldots = \frac{X_n a_n}{\alpha_n}.$$

R. VYTHYNATHASWAMY.

Note on a Formula in Solid Geometry.

In the issue of the *Journal* for February last, Mr. Gulasekharam gave under this title a neat solution of the problem of calculating the angles between the lines in which a cone is cut by a plane through its vertex. The following method of dealing with the same problem is neither original nor new; I myself had it from Dr. Bromwich several years ago, and there is an echo of it in § 7°34 of my *Multilinear Functions of Direction*. The method deserves to be more widely known. In what follows I apply it to the case in which the axes of reference are oblique, partly in order to show how slight is the complication so produced. Also I enunciate the extension to a vector frame of any kind, which renders the result available in the theory of curvilinear co-ordinates.

The notation is that which I have used elsewhere: the angles α, β, γ are the angles of the frame, which Bell, Frost, and Salmon denote by λ, μ, ν.

The axes being oblique, to find the angles between two directions whose ratios (x_1, y_1, z_1), (x_2, y_2, z_2) *are given by the pair of equations*

$$u\,x + v\,y + w\,z = 0, \qquad \dots \quad (1)$$

$$a\,x^2 + b\,y^2 + c\,z^2 + 2f\,yz + 2g\,zx + 2h\,xy = 0. \qquad \dots \quad (2)$$

The proportions $u : v : w$ are those of the *cosines* of a direction perpendicular to both the required directions, and the actual cosines of this direction are $(u/q, v/q, w/q)$, where

$$q^2 = -\,\mathrm{Y}^{-2} \begin{vmatrix} 1 & \cos\gamma & \cos\beta & u \\ \cos\gamma & 1 & \cos\alpha & v \\ \cos\beta & \cos\alpha & 1 & w \\ u & v & w & 0 \end{vmatrix}; \qquad \dots \quad (3)$$

in this formula, Y is the sine of the frame, given by

$$\mathrm{Y}^2 = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}. \qquad \dots \quad (4)$$

If θ is an angle from the direction whose *ratios* are (x_1, y_1, z_1) to the direction whose *ratios* are (x_2, y_2, z_2) round the direction whose *cosines* are $(u/q, v/q, w/q)$, then

$$(u, v, w) \sin\theta = \mathrm{Y}q \begin{Vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{Vmatrix}. \qquad \dots \quad (5)$$

Let λ, μ, ν be any three numbers, and write

$$\left.\begin{aligned} x_1 \lambda + y_1 \mu + z_1 \nu &= k_1, \\ x_2 \lambda + y_2 \mu + z_2 \nu &= k_2. \end{aligned}\right\} \qquad (6)$$

From (6), identically,

$$k_1 x_2 - k_2 x_1 = (z_1 x_2 - z_2 x_1)\,\nu - (x_1 y_2 - y_2 x_1)\mu,$$

whence from (5),

$$\mathrm{Y}q\,(k_1 x_2 - k_2 x_1) = (v\,\nu - w\mu)\sin\theta\,; \ldots \qquad \ldots \quad (7)$$

similarly

$$\mathrm{Y}q\,(k_1 y_2 - k_2 y_1) = (w\,\lambda - u\,\nu)\sin\theta,$$
$$\mathrm{Y}q\,(k_1 z_2 - k_2 z_1) = (u\,\mu - v\lambda)\sin\theta. \qquad \ldots \quad (8)$$

Now consider the determinant

$$\begin{vmatrix} a & h & g & u & \lambda \\ h & b & f & v & \mu \\ g & f & c & w & \nu \\ u & v & w & 0 & 0 \\ \lambda & \mu & \nu & 0 & 0 \end{vmatrix}$$

Expanded in terms of λ, μ, ν, this is

$$-(\mathrm{A}\,\lambda^2 + \mathrm{B}\,\mu^2 + \mathrm{C}\,\nu^2 + 2\,\mathrm{F}\,\mu\,\nu + 2\,\mathrm{G}\,\nu\,\lambda + 2\,\mathrm{H}\,\lambda\,\mu)$$

where the co-efficients are the co-factors of a, b, c, f, g, h in

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

But the five-rowed determinant may be expanded otherwise as

$$a\,(v\,\nu - w\,\mu)^2 + \ldots\ldots + \ldots\ldots + 2f\,(w\,\lambda - u\,\nu)\,(u\,\mu - v\,\lambda)$$
$$+ \ldots\ldots\ldots + \ldots\ldots\ldots,$$

and this expression, on substitution from (7) and (8) takes the form

$$\{\,a\,(k_1 x_2 - k_2 x_1)^2 + \ldots\ldots + \ldots\ldots\ldots + 2f\,(k_1 y_2 - k_2 y_1)$$
$$(k_1 z_2 - k_2 z_1) + \ldots\ldots + \ldots\ldots\,\}\ \mathrm{Y}^2 q^2 \operatorname{cosec}^2\theta.$$

Since (x_1, y_1, z_1) and (x_2, y_2, z_2) satisfy (2), the terms multiplied by k_1^2 and k_2^2 vanish, and the whole expression reduces to

$$-2\,k_1 k_2\,\mathrm{K}\,\mathrm{Y}^2 q^2 \operatorname{cosec}^2\theta,$$

where

$$K = a\,x_1x_2 + \ldots\ldots + f\,(y_1z_2 + z_1y_2) + \ldots\ldots + \ldots\ldots\ldots$$

Replacing k_1, k_2, by their values from (6), we have the equality

$$A\,\lambda^2 + B\,\mu^2 + C\nu^2 + 2F\,\mu\nu + 2G\,\nu\,\lambda + 2\,H\,\lambda\,\mu$$
$$= 2\,(x_1\,\lambda + y_1\,\mu + z_1\,\nu)\,(x_2\,\lambda + y_2\,\mu + z_2\,\nu)\;K\;Y^2q^2\,\mathrm{cosec}^2\,\theta,$$

true for *all* sets of values of λ, μ, ν. Hence, equating co-efficients, we have

$$\frac{x_1\,x_2}{A} = \ldots = \ldots = \frac{y_1\,z_2 + z_1\,y_2}{2\,F} = \ldots\ldots = \ldots\ldots = \frac{\sin^2\theta}{2\,K\,Y^2q^2}.$$

But the common value of the ratios is equal identically to

$$\frac{a\,x_1x_2 + \ldots + \ldots + f\,(y_1z_2 + z_1y_2) + \ldots\ldots + \ldots\ldots}{a\,A + \ldots + \ldots + 2f\,F + \ldots + \ldots\ldots},$$

that is, to
$$\frac{K}{2\Delta},$$

where Δ is the value of the four-rowed determinant, and is equal also to

$$\frac{x_1x_2 + \ldots + \ldots + (y_1z_2 + z_1y_2)\cos\alpha + \ldots + \ldots\ldots}{A + \ldots + \ldots + 2F\cos\alpha + \ldots + \ldots\ldots},$$

in which the value of the numerator is $\cos\theta$. Thus

$$\frac{\cos\theta}{A + B + C + 2F\cos\alpha + 2G\cos\beta + 2H\cos\gamma} = \frac{\sin^2\theta}{2KY^2q^2} = \frac{K}{2\Delta}.$$

From the equality of the last two fractions, each fraction is equal to

$$\pm\frac{\sin\theta}{2\sqrt{(\Delta Y^2q^2)}},$$

and so we have finally

$$\cot^2\theta = (A + B + C + 2F\cos\alpha + 2G\cos\beta + 2H\cos\gamma)^2 / 4\Delta Y^2q^2.$$

There is something artificial about the method, and yet it does relate the group of combination x_1x_2,......to the groups of co-factors A,......in a single operation, and it does avoid complications when the axes are oblique. Mr. Gulasekharam's determination of the equalities

$$\frac{x_1x_2}{A} = \ldots = \ldots = \frac{y_1x_2 + x_1y_2}{2F} = \ldots\ldots = \ldots\ldots$$

is independent of the assumption that the axes are rectangular, and so also is the deduction that the square of these fractions is equal to each of the fractions

$$\frac{(y_1z_2 - z_1y_2)^2}{4\Delta u^2},\quad \frac{(z_1x_2 - x_1z_2)^2}{4\Delta v^2},\quad \frac{(x_1y_2 - y_1x_2)^2}{4\Delta w^2},$$

but with oblique axes the formula for $\sin^2\theta$ involves the products such as

$$(z_1x_2 - x_1z_2)(x_1y_2 - y_1x_2)..$$

To complete the proof on his lines we have to use the equalities

$$\frac{y_1z_2 - z_1y_2}{u} = \frac{z_1x_2 - x_1z_2}{v} = \frac{x_1y_2 - y_1x_2}{w},$$

which come immediately from (1). Then since*

$$\sin^2\theta = -\frac{1}{a}\begin{vmatrix} 1 & \cos\gamma & \cos\beta & y_1z_2 - z_1y_2 \\ \cos\gamma & 1 & \cos\alpha & z_1x_2 - x_1z_2 \\ \cos\beta & \cos\alpha & 1 & x_1y_2 - y_1x_2 \\ y_1z_2 - z_1y_2 & z_1x_2 - x_1z_2 & x_1y_2 - y_1x_2 & 0 \end{vmatrix}$$

it follows as a matter of pure algebra that the square of the fractions is equal to a fraction whose numerator is $\sin^2\theta$ and whose denominator is -4Δ times

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & u \\ \cos\gamma & 1 & \cos\alpha & v \\ \cos\beta & \cos\alpha & 1 & w \\ u & v & w & 0 \end{vmatrix}$$

The extension, either of the analysis or of the final formula, to a vector frame, is immediate, and I need only enunciate the result:

In a vector frame with fundamental magnitudes L, M, N, P, Q, R *if* θ *is an angle between two vectors whose co-efficients satisfy simultaneously the two equations*

$$u\xi + v\eta + w\zeta = 0,$$

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0,$$

then $\cot^2\theta$ *is the value of the fraction whose denominator is the negative of four times the product of the two determinants*

$$\begin{vmatrix} L & R & Q & u \\ R & M & P & v \\ Q & P & N & w \\ u & v & w & 0 \end{vmatrix} \qquad \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

and whose numerator is the square of

$$LA + MB + NC + 2PF + 2QG + 2RH,$$

where A, B, C, F, G, H *are the co-factors of* a, b, c, f, g, h *in the latter of the determinants.*

E. H. Neville.

* See for example my *Prolegomena*, p. 138.

Solutions.

Question 940.

(R. SRINIVASAN, M. A : M. R. A. S.) :—Show that

$$\int_0^{\phi} \frac{x^n \log x}{1+x^2}\, dx = \frac{\pi^2}{4} \sec \frac{n\pi}{2} \tan \frac{n\pi}{2} .$$

Solution by S. Audinarayanan.

[*N. B.*—The upper limit of the integral should be ∞.]

Consider the integral

$$f(x) = \int_0^{\infty} \frac{x^n\, dx}{1+x^2} .$$

The integrand is continuous and hence could be differentiated under the sign of integration.

If $0 < n < 1$, we have

$$\int_0^{\infty} \frac{x^n\, dx}{1+x^2} = \frac{\pi}{2 \cos \frac{n\pi}{2}} .$$

[Williamson : *Integral Calculus*, page 142.]

Hence differentiating we have

$$\frac{df(x)}{dn} = \int_0^{\infty} \frac{x^n \log x}{1+x^2}\, dx.$$

$$\frac{d}{dn}\left[\frac{\pi}{2 \cos \frac{n\pi}{2}}\right] = \frac{\pi^2}{4} \sec. \frac{n\pi}{2} \tan \frac{n\pi}{2} .$$

Hence $$\int_0^{\infty} \frac{x^n \log x}{1+x^2}\, dx = \frac{\pi^2}{4} \sec \frac{n\pi}{2} \tan \frac{n\pi}{2} .$$

Question 1150.

(MARTYN M. THOMAS, M. A.) :—If ρ and ρ_n be the corresponding radii of curvature of the curve $p = f(r)$ and its n^{th} negative pedal, show that

$$\frac{nr^2 - (n-1)p\rho_n}{(n+1)r^2 - np\rho_n} = \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} - \ldots - \tfrac{1}{2} - \frac{p\rho_n}{r^2}$$

to $(n+1)$ fractions

$$= \frac{r^{n-2}\rho}{p^{n-1}}.$$

Solution by M. V. Ramakrishnan and K. Satyanarayana.

With the usual notation

$$r = \frac{p_n{}^n}{r_n{}^{n-1}}\,;\ \text{and } p = \frac{p_n{}^{n+1}}{r_n{}^n}.$$

Hence the n^{th} negative pedal of $p = f(r)$ is

$$r^n = \frac{p^{n+1}}{f\left(\dfrac{p^n}{r^{n-1}}\right)}, \text{ dropping the suffixes.}$$

or

$$r^n \cdot f\left(\frac{p^n}{r^{n-1}}\right) = p^{n+1}$$

Differentiating with respect to p,

$$r^n f' \left\{ \frac{r^{n-1}\, np^{n-1} - (n-1)r^{n-2}p^n \dfrac{dr}{dp}}{r^{2(n-1)}} \right\} + nr^{n-1}\frac{dr}{dp}\cdot f = (n+1)p^n.$$

$$i.e. \quad r^n f' \left\{ \frac{nr^2 - (n-1)p\rho_n}{r^{n+1}} \right\} p^{n-1} + nr^{n-2}\rho_n \frac{p^{n+1}}{r^n} = (n+1)p^n$$

$$\therefore \quad \frac{nr^2 - (n-1)p\rho_n}{r}\, p^{n-1} f' = \frac{(n+1)r^2 - np\rho_n}{r^2}\, p^n.$$

$$\therefore \quad \frac{nr^2 - (n-1)p\rho_n}{(n+1)r^2 - np\rho_n} = \frac{p}{rf'}.$$

But r of the original curve $= \dfrac{p_n{}^n}{r_n{}^{n-1}}$, so that

$$\rho = \frac{p_n{}^n}{r_n{}^{n-1}} \cdot \frac{1}{f'} = \frac{p^n}{r^{n-1}} \cdot \frac{1}{f'}, \text{ dropping the suffixes.}$$

Therefore we have

$$\frac{nr^2 - (n-1)\,p\rho_n}{(n+1)r^2 - np\rho_n} = \frac{p}{r}\cdot\frac{r^{n-1}\rho}{p^n}$$

$$= \frac{r^{n-2}\rho}{p^{n-1}}.$$

Now it is easy to see that the $(n-1)^{\text{th}}$ and the n^{th} convergents of

$$\tfrac{1}{2}-\tfrac{1}{2}-\ldots-\tfrac{1}{2}-\frac{p\rho_n{}^n}{r^2} \text{ to } (n+1) \text{ fractions}$$

are respectively

$$\frac{n-1}{n} \text{ and } \frac{n}{n+1},$$

so that the $(n+1)^{\text{th}}$ convergent $= \dfrac{nr^2-(n-1)\,p\rho_n}{(n+1)r^2 - np\rho_n}$.

Therefore

$$\frac{nr^2-(n-1)\,p\rho_n}{(n+1)r^2-np\rho_n} = \tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}-\ldots-\tfrac{1}{2}-\frac{p\rho_n}{r^2}$$

to $n+1$ fractions

$$= \frac{r^{n-2}\rho}{p^{n-1}}$$

Question 1153.

(R. VAIDYANATHASWAMI) :—Find three collinear points on a cusped cubic the tangents at which are concurrent. Prove that the line containing the three points passes through a fixed point and the point of concurrence of the tangents traces a fixed line.

Solution by the Proposer.

Let the parameter of the cusp be zero and of the point of inflexion, ∞. The complete system of collinear triads has the foci 0, 0, ∞. Hence Γ contains the triads (0, 0, 0) and (∞, ∞, ∞). Reciprocally the complete system Γ' of points of contact of concurrent tangents has the foci (∞, ∞, 0) and hence contains the triads (∞, ∞, ∞), (0, 0, 0). Hence the common system G of Γ and Γ' is the focal system whose foci are (0, ∞). We can interpret G in two ways according as we regard it as a sub-system of Γ or Γ'. In the former case G is the system of intersections of the cubic with the lines which are concurrent with the lines (0, 0, 0), (∞, ∞, ∞) *i.e.* with the cuspidal and inflexional tangents. In the latter case G is the system of contacts of tangents from points collinear with the points (0, 0, 0), (∞, ∞, ∞) *i.e.* with the cusp and inflexion.

Question 1154.

(R. VAIDYANATHASWAMI) :—Find all the conics which reciprocate a cusped cubic into itself.

Solution by the Proposer.

Any conic which reciprocates a cusped cubic into itself must reciprocate the cusp into the inflexional tangent and the inflexion into the cuspidal tangent. Hence it has the inflexion-cuspidal Δ for a self-conjugate Δ. Further if it reciprocates the point p on the curve into the tangent at q, then pq belong to an involution on the curve containing the cusp-inflexion pair. Hence if P, Q, be the fixed points of this involution, PQ must pass through the inflexion. It is also clear that the reciprocating conic has double-contact with the cubic at P,Q *i.e.*, at the extremities of some chord of the curve through the inflexion. These conditions completely define the reciprocating conics.

Question 1208.

(G. V. T.) :—Given an angle of a triangle in position and the distance between the circum-centre and the incentre in magnitude and direction, construct the triangle.

Solutions (1) *by V. Satyanarayana,* (2) *K. Srinivasaraghavan, and* (3) *by X. Y. Z.*

(1) We know that, if AI meet the circumcircle in P,

$$AI \, . \, IP = 2Rr.$$

i.e.

$$AI : r = 2R : IP.$$

But AI : r is a given ratio, since the angle A is given. Hence, the ratio 2R : IP is given.

Thus in the triangle SIP, SI being given in direction and magnitude, and also the ratio SP : IP as well as the angle SIP, the construction of the triangle SIP follows at once. The rest can be deduced easily.

(2) Let DEF, XYZ be the feet of the $\perp$rs from S and I on the sides BC, CA and AB of the ΔABC. Then DX $= \frac{1}{2}(b - c)$, EY $= \frac{1}{2}(c - a)$ and FZ$=\frac{1}{2}(a - b)$, attention being paid to signs. The sum of any two is equal to the third with the sign changed. Since the direction and the distance of SI are given, and since the angle BAC is fixed, we know that FZ and EY are fixed in direction and magnitude. Hence we derive the length DX, and as the direction of SI is fixed and since SI is of constant length, the direction of SD and IX are fixed.

Thus the direction of BC is fixed. Hence the angles B and C are known and the construction easily follows.

(3) If XYZ be the ex-central triangle, its N. P. centre and ortho-centre will be S and I respectively. Hence the problem reduces to constructing the triangle XYZ in which the distance between the N. P. centre and (therefore also the circum-centre) and ortho-centre is given in position and magnitude and also the angle X is given.

Now, if the circum-centre of XYZ be O,

$$OX : XI = R : 2R\cos X = 1 : 2\cos X.$$

$$\therefore \quad OX : XI = \text{a given quantity}.$$

Hence OI being given and the angle OIX, the triangle OXI is at once constructed. The triangle ABC is the pedal triangle of XYZ.

Question 1214.

(G. V. Telang) :—If the centre of a pedal circle of any point P with respect to a triangle ABC lies on a line $l\alpha + m\beta + n\gamma = 0$, show that the locus of P is the cubic,

$$\Sigma(mc + nb)\,\alpha\,(\beta^2 + \gamma^2 + 2\beta\gamma\cos A) = 0.$$

If the centre lies on the N. P. circle of the triangle ABC, the locus of the point P is

$$\Sigma \frac{a^3}{\alpha(\beta^2 + \gamma^2 + 2\beta\gamma\cos A)} = 0.$$

Solution by M. V. Ramakrishnan and N. Sundaram Iyer.

The pedal circle of P is the auxiliary circle of the conic inscribed in $\triangle$ABC. Hence the co-ordinate of the centre of the circle can easily be obtained. If $\sqrt{\lambda\alpha} + \sqrt{\mu\beta} + \sqrt{\nu\gamma} = 0$ be the in-conic, the co-ordinates of the centre are,

$$(b\nu + c\mu) : (c\lambda + a\nu) : (a\mu + \lambda b).$$

Hence if the centre lies on $l\alpha + m\beta + n\gamma = 0$, we have

$$l\,(b\nu + c\mu) + m(c\lambda + a\nu) + n\,(a\mu + \lambda b) = 0,$$

or $$\lambda(mc + nb) + \mu(na + lc) + \nu(lc + ma) = 0.$$

Now the co-ordinates of the centre referred to the medial triangle of ABC are

$$\frac{\lambda}{a^2} : \frac{\mu}{b^2} : \frac{\nu}{c^2}.$$

From Vol. XIII, 6 (*J. I. M. S.*), page 213, the co-ordinates of the centre referred to the medial triangle of ABC are

$$\operatorname{cosec} P\hat{A}B \operatorname{cosec} P\hat{A}C : \operatorname{cosec} P\hat{B}C \operatorname{cosec} P\hat{B}A : \operatorname{cosec} P\hat{C}A \operatorname{cosec} P\hat{C}B$$

Hence if P is $(\alpha'\beta'\gamma')$

$$\frac{\frac{\lambda}{a^2}}{\alpha'.PA^2} = \frac{\frac{\mu}{b^2}}{\beta'.PB^2} = \frac{\frac{\nu}{c^2}}{\gamma'.PC^2}.$$

or

$$\frac{\lambda}{\alpha'(\beta'^2+\gamma'^2+2\beta'\gamma'\cos A)} = \frac{\mu}{\beta'(\gamma'^2+\alpha'^2+2\gamma'\alpha'\cos B)}$$

$$= \frac{\nu}{\gamma'(\alpha'^2+\beta'^2+2\alpha'\beta'\cos C)}.$$

Hence the equation to the locus of P is

$$\Sigma\,(mc+nb)\,\alpha\,(\beta^2+\gamma^2+2\beta\gamma\cos A) = 0.$$

If the centre lies on the N. P. circle whose equation is

$$\Sigma \frac{a^2}{b\beta + c\gamma - a\alpha} = 0,$$

we have

$$\Sigma \frac{a^2}{\lambda bc} = 0,$$

or

$$\Sigma \frac{a^3}{\lambda} = 0,$$

so that the locus of P is

$$\Sigma \frac{a^3}{\alpha(\beta^2+\gamma^2+2\beta\gamma\cos A)} = 0.$$

Question 1216.

(F. H. V. GULASEKARAM) :—The absolute normal co-ordinates of the centre of an inconic of the triangle of reference are α, β, γ. Prove that the squares of the semi-axes are given by the equation

$$a\sqrt{r^2-\alpha^2} + b\sqrt{r^2-\beta^2} + c\sqrt{r^2-\gamma^2} = 0.$$

Solution (1) *by K. C. Shah, M.A.,*
and (2) *by N. Sundaram Iyer and the proposer.*

(1) By elementary geometry it can be shewn that the square on the perpendicular from the centre of a conic to any tangent is equal to the sum of the squares on the projections of the semi-axes on that perpendicular. Hence if the perpendiculars from the centre of the given in-conic on the tangents BC, CA, AB make angles θ_1 θ_2 θ_3 respectively with the major axis, we have

$$\alpha^2 = \rho_1^2 \cos^2 \theta_1 + \rho_2^2 \sin^2 \theta_1$$

$$\beta^2 = \rho_1^2 \cos^2 \theta_2 + \rho_2^2 \sin^2 \theta_2$$

$$\gamma^2 = \rho_1^2 \cos^2 \theta_3 + \rho_2^2 \sin^2 \theta_3,$$

where $2\rho_1$ and $2\rho_2$ are the major and minor axes respectively.

$$\therefore \quad \rho_1^2 - \alpha^2 = (\rho_1^2 - \rho_2^2) \sin^2 \theta_1.$$

$$\therefore \quad \Sigma(\rho_1^2 - \alpha^2)^{\frac{1}{2}} \sin(\theta_2 \sim \theta_3)$$

$$= (\rho_1^2 - \rho_2^2)^{\frac{1}{2}} \Sigma \sin \theta_1 \sin(\theta_2 \sim \theta_3) = 0.$$

Also $\quad \theta_2 \sim \theta_3 = \pi - A$, &c.

$$\therefore \quad \Sigma(\rho_1^2 - \alpha^2)^{\frac{1}{2}} \sin A = 0.$$

Similarly $\quad \Sigma(\rho_2^2 - \alpha^2)^{\frac{1}{2}} \sin A = 0.$

Hence ρ_1^2 and ρ_2^2 are the roots of

$$a\sqrt{r^2 - \alpha^2} + b\sqrt{r^2 - \beta^2} + c\sqrt{r^2 - \gamma^2} = 0.$$

(2) Let P, P′ be the foci, and DD′, EE′, FF′ be the projections of PP′ on the sides BC, CA, AB, respectively.

Then

$$DD' = 2\sqrt{r^2 - \alpha^2},$$

$$EE' = 2\sqrt{r^2 - \beta^2},$$

$$FF' = 2\sqrt{r^2 - \gamma^2},$$

where r is the semi-major axis, and

$$a \cdot DD' + b \cdot EE' + c \cdot FF' = 0.$$

(*J. I. M. S.*, Vol. V, p. 75).

Hence $\quad a\sqrt{r^2 - \alpha^2} + b\sqrt{r^2 - \beta^2} + c\sqrt{r^2 - \gamma^2} = 0,$

which gives the squares of the semi-axes.

Questions for Solution.

1290. (K. J. SANJANA and I. B. MUKHERJI) :—Examine whether the following two problems are identical or not, and solve the first of them :—

"Find the size of a cube which will stop up a tube of uniform bore, the section of which is a regular hexagon whose sides are given." TODHUNTER.

"Show that, in general, a plane which cuts the six faces of a cube cuts them in a hexagon whose opposite sides are parallel, and show how to make the section a regular hexagon." DAVISON.

1291. (MARTYN M. THOMAS, M.A.) :—If the equation of a curve in multiple angular co-ordinates, be

$$\theta_1 + \theta_2 + \theta_3 + \ldots\ldots \theta_n = \text{a constant},$$

show that the equation of its orthogonal trajectory, in multiple polar co-ordinates, is

$$r_1 r_2 r_3 \ldots\ldots r_n = \text{a constant}.$$

[*Particular Case* :—The orthogonal trajectories of rectangular hyperbolas are Cassini's ovals.]

1292. (V. RAMASWAMY AIYAR) :—If a and b be positive and unequal, show that

$$\frac{\frac{1}{2}(a^2 - b^2)}{a - b} < \frac{\frac{1}{3}(a^3 - b^3)}{\frac{1}{2}(a^2 - b^2)} < \frac{\frac{1}{4}(a^4 - b^4)}{\frac{1}{3}(a^3 - b^3)} < \quad \ldots\ldots$$

without limit.

(2) Show further that

$$\left\{ \frac{a^m - b^m}{m(a - b)} \right\}^{\frac{1}{m-1}}$$

always increases as m increases.

(3) Hence or otherwise show that

$$\frac{a^m - b^m}{a - b} \gtrless m\,(ab)^{\frac{m-1}{2}}, \text{ according as } m(m^2 - 1) \gtrless 0;$$

and that

$$\frac{a^m - b^m}{a - b} \gtrless m \left(\frac{a + b}{2} \right)^{m-1}, \text{ according as } m(m - 1)(m - 2) \gtrless 0.$$

(4) Show also that

$$\sqrt{ab} < \frac{a - b}{\log a - \log b} < \frac{a^{\frac{a}{a-b}} \cdot b^{\frac{b}{b-a}}}{e} < \frac{a + b}{2}.$$

1293. (MARTYN M. THOMAS, M.A.):—If in bipolar co-ordinates, the equation of a family of curves be $f(r, r') = c$, prove that the differential equation of the orthogonal trajectories is

$$r \frac{\partial f}{\partial r} d\theta = r' \frac{\partial f}{\partial r'} d\theta'.$$

1294. (M. BHEEMASENA RAO and M. VENKATARAMAN):—

Show that :—

$$(a) \quad \begin{vmatrix} \sin\theta & \sin 3\theta & \sin 5\theta & . \; . & \sin(2n-1)\theta \\ \sin 2\theta & \sin 3.2\theta & \sin 5.2\theta & . \; . & \sin(2n-1)2\theta \\ \sin 3\theta & \sin 3.3\theta & \sin 5.3\theta & . \; . & \sin(2n-1)3\theta \\ . & . & . & . \; . & . \\ . & . & . & . \; . & . \\ \sin n\theta & \sin 3.n\theta & \sin 5.n\theta & . \; . & \sin(2n-1)n\theta \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} \cdot \left(\frac{2n+1}{4}\right)^{\frac{n}{2}}, \text{ if } \theta = \frac{\pi}{2n+1}.$$

$$\text{or} \quad (-1)^{\frac{n(n-1)}{2}} \frac{1}{\sqrt{2}} \left(\frac{n+1}{2}\right)^{\frac{n-1}{2}}, \text{ if } \theta = \frac{\pi}{2n+2}.$$

$$(b) \quad \begin{vmatrix} \sin 2\theta & \sin 4\theta & \sin 6\theta & . \; . & \sin 2n\theta \\ \sin 4\theta & \sin 8\theta & \sin 12\theta & . \; . & \sin 4n\theta \\ \sin 6\theta & \sin 12\theta & \sin 18\theta & . \; . & \sin 6n\theta \\ . & . & . & . \; . & . \\ . & . & . & . \; . & . \\ \sin 2n\theta & \sin 4n\theta & \sin 6n\theta & . \; . & \sin 2n^2\theta \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} \cdot \left(\frac{n+1}{4}\right)^{\frac{n}{2}}, \text{ if } \theta = \frac{\pi}{2n+1};$$

$$\text{or} \quad (-1)^{\frac{n(n-1)}{2}} \cdot \left(\frac{n+1}{2}\right)^{\frac{n}{2}}, \text{ if } \theta = \frac{\pi}{2n+2}.$$

1295. (M. BHIMASENA RAO):—

Solve :—

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 4x(1+x^2)\frac{dy}{dx} + (1+2x^2)y = 0.$$

LIST OF JOURNALS RECEIVED BY THE SOCIETY.

1 Messenger of Mathematics
2 Quarterly Journal of Mathematics
3 Mathematical Gazette
4 The Annals of Mathematics
5 American Journal of Mathematics
6 Bulletin of the American Mathematical Society
7 Transactions of the American Mathematical Society
8 Monthly Notices of the Royal Astronomical Society
9 Proceedings of the Royal Society of London
10 The Philosophical Magazine and Journal of Science
11 Astrophysical Journal
12 Crelle's Journal
13 L'intermediaere des Mathematiciens
14 Mathematische Annalen
15 Philosophical Transactions of the Royal Society of London
16 Acta Mathematica
17 Popular Astronomy
18 Proceedings of the Edinburgh Mathematical Society
19 Proceedings of the London Mathematical Society
20 Mathematics Teacher
21 Bulletin of the Calcutta Mathematical Society
22 The Tohoku Mathematical Journal
23 Nature
24 The American Mathematical Monthly
25 Proceedings of the Benares Mathematical Society

The Indian Mathematical Society

(Founded in 1907 for the Advancement of Mathematical Study and Research in India.)